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THE METHOD OF BOUNDARY EQUATIONS OF THE HAMMERSTEIN-TYPE FOR CONTACT PROBLEMS OF THE THEORY OF ELASTICITY WHEN THE REGIONS OF CONTACT ARE NOT KNOWN*

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Novel formulations are given for the classical and non-classical, three-dimensional contact problems. The inequality-type constraints do not appear in the formulations as they do in the method of varational inequalities /1–8/ and in existing formulations of the contact problems /9– 13/. The complete system of equations of a contact problem consists of one boundary, equation of the Hammerstein-type and the usual equations of equilibrium for the compressive force and moments acting on the bodies. If the mutual rotations of the bodies and their closeness are known, the solution of the boundary Hammerstein-type equation readily yields the contact pressure and region of contact.

By formulating the problem in this manner and using modern methods of the theory of operator equations we can investigate the existence and uniqueness of the solutions and some of their properties in very general cases (e.g. those of the multiconnectivity of the regions of contact sought). Moreover, the possibility arises of solving the problem using existing methods of solving Hammerstein-type equations /14–17/. Two types of problems, one of them classical, are used to study the correctness of the formulation of the contact problem.

1. The classical contact problem. Let us consider the case when the contact problem can be reduced to determining, in the half-space $z > 0$, the harmonic function $u(M) = u(x, y, z)$ $(u(M) = O(r^{-1})$ as $r \to \infty$, $r = \sqrt{x^2 + y^2 + z^2})$, and a plane closed region $S \subset E_2 = \{z = 0\}$ from the conditions

$$\begin{aligned} &2\pi\lambda u(M) = g(M);\ u_z'(M) \geqslant 0,\ M \in S \\ &2\pi\lambda u(M) > g(M);\ u_z'(M) = 0,\ M \in (E_2 \setminus S) \\ &(g(M) \in C(E_2),\ \lambda = \mathrm{const} > 0) \end{aligned} \qquad (1.1)$$

We assume that a bounded region $\Omega_0 = \{M{:}\ g(M) > 0\}$, $g(M) \leqslant 0$ with $M \notin \Omega_0$ exists (the region Ω_0 can be multiply connected).

Such a formulation of the problem corresponds to frictionless imbedding of a stamp in an elastic half-space /9–13/, provided that the settlement of the stamp and its rotation are both known, i.e. that the function $g(M) = h_1 + h_2x + h_3y - f(x, y)$ is known. The function $f(x, y)$ determines the geometry of the stamp.

If we introduce the potential of the simple layer of density $p(M)$, $M \in S$, then we have for $z = 0$

$$u(M) = \frac{1}{2\pi}\int_S K(M, N)\, p(N)\, dS_N;\quad K(M, N) = [(x-\xi)^2 + (y-\eta)^2]^{-1/2};\quad M(x, y),\quad N(\xi, \eta)$$

and problem (1.1) becomes equivalent to the problem of determining the contact pressure $p(M)$, and the regions of contact S from the system

$$\begin{aligned} &\lambda\int_S K(M, N)\, p(N)\, dS_N = g(M);\quad p(M) \geqslant 0,\quad M \in S \\ &\lambda\int_S K(M, N)\, p(N)\, dS_N > g(M);\quad p(M) = 0,\quad M \in (\Omega \setminus S) \end{aligned} \qquad (1.2)$$

where Ω is an arbitrarily bounded region containing the closure Ω_0. Clearly, $S \in \bar{\Omega}_0$.

Let us introduce the positively homogeneous bounded operators Q and Q^-, placing in the functions $v(M), M \in \Omega$ in correspondence with the functions $v^+(M)$ and $v^-(M)$, $M \in \Omega$, according to the rules

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$$v^+(M) = Q(v(M)) = \sup\{v(M),\ 0\} \tag{1.3}$$
$$v^-(M) = Q^-(v(M)) = \inf\{v(M), 0\};\ v(M) = v^+(M) + v^-(M)$$

Let us investigate, for the unknown function $v(M)$, the non-linear equation

$$\mu Q^-(v(M)) + \lambda \int_\Omega K(M,N)\, Q(v(N))\, dS_N = g(M);\quad M, N \in \Omega$$

which we shall write for convenience in the operator form as

$$\mu Q^- v + \lambda K Q v = g \tag{1.4}$$

The parameter $\mu > 0$ can take any value. The dependence of the solution of Eq.(1.4) on μ will be shown later (Theorem 5 and its corollary).

Elementary transformations reduce Eq.(1.4) to a Hammerstein-type equation and can be written in the following equivalent forms:

$$v = \mu^{-1} g + BQv,\ \ w = BFw$$
$$(B = E - \lambda\mu^{-1}K,\ \ w = \mu v - g,\ \ Fw = Q(w + g))$$

where B is a linear operator and E is an identity operator.

Theorem 1. If v^* is a solution of Eq.(1.4), then $(p = Qv^*,\ \ S = \{M : v^* \geqslant 0\})$ is a solution of system (1.2) and $S \neq \varnothing$ when $\Omega_0 \neq \varnothing$; conversely, if (p, S) is a solution of system (1.2), then the function

$$v^* = \mu^{-1} g + p - \lambda\mu^{-1} K p,\ \ M \in \Omega \tag{1.5}$$

is a solution of (1.4). The region S can be multiply connected.

Proof. First we shall show that $S \neq \varnothing$ if $\Omega_0 \neq \varnothing$. Let us assume the opposite. Then (1.4) implies the inequality $g < 0$ which contradicts the existence of $\Omega_0 \neq \varnothing$.

Let v^* be a solution of (1.4). When $M \in S$, (1.3), (1.4) imply the relations $p = Qv^* \geqslant 0$, $\lambda K p = g$. If $M \notin S$, then $v^* < 0$, $\mu Q^- v^* + \lambda K Q v^* = g$ and $\lambda K p > g$ and this proves the right-hand side of the theorem.

Now let (p, S) be a solution of system (1.2). When $M \in S$, the equality $v^* = p$ follows from (1.2), (1.5). If $M \in (\Omega \setminus S)$, then we have $v^* = \mu^{-1} g - \lambda\mu^{-1} K p < 0$ and we can write (1.5) as:

$$v^* = \mu^{-1} g + Qv^* - \lambda\mu^{-1} K Q v^*,\ M \in \Omega$$

i.e. v^* is a solution of (1.4).

Hence, to solve the contact problem (1.2), it is sufficient to find the solution v^* of Eq.(1.4), since $p = Qv^*$ and $S = \{M\colon v^* \geqslant 0\}$. Therefore from now on we shall concern ourselves with Eq.(1.4).

In addition to (1.4) we shall consider, in $L_2(\Omega)$, a regularized boundary condition

$$\varepsilon Q v + \mu Q^- v + \lambda K Q v = g \tag{1.6}$$

with parameter $\varepsilon > 0$. We will write Eq.(1.6) in the form

$$v = \mu^{-1} g + B_\varepsilon Q v\ \ (B_\varepsilon = (1 - \varepsilon\mu^{-1}) E - \lambda\mu^{-1} K) \tag{1.7}$$

and study it for such fairly large values of μ that when $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 = \text{const} > 0$, the following inequality holds:

$$\| B_\varepsilon \| = \sup_k | 1 - \varepsilon\mu^{-1} - \lambda\mu^{-1}\lambda_k^{-1} | = q < 1$$

Here $\lambda_k > 0$ are the characteristic values of the operator K (the norm of the operator is defined in /17/, p.191).

Then (by virtue of the principle of compressed mappings applied to (1.7)) Eq.(1.6) has a solution $v_\varepsilon \in L_2(\Omega)$ for all $\varepsilon \in (0, \varepsilon_0]$. Since the kernel $K(M, N)$ has a weak singularity, $v_\varepsilon \in C(\Omega)$.

Let us introduce the notation

$$\Omega_\varepsilon^+ = \{M : v_\varepsilon \geqslant 0\},\quad \Omega_\varepsilon^- = \{M : v_\varepsilon < 0\},\quad (a, b) = \int_\Omega a(t)\, b(t)\, dt$$

The differential properties of the solutions v_ε of (1.6) are established by the following lemma.

Lemma. If $g' \in L_p(\Omega)$, $1 \leqslant p \leqslant \infty$, then $v_\varepsilon' \in L_p(\Omega)$; if $g' \in C(\Omega)$, then $v_\varepsilon^{+\prime} \in C(\Omega_\varepsilon^+)$, $v_\varepsilon^{-\prime} \in C(\Omega_\varepsilon^-)$.

Proof. We have for $M \in \Omega_\varepsilon^+$

$$v_\varepsilon(M) = \varepsilon^{-1} g(M) - \lambda\varepsilon^{-1} \int_{\Omega_\varepsilon^+} K(M, N)\, v_\varepsilon(N)\, dS_N \tag{1.8}$$

$$v_\varepsilon'(M) = \varepsilon^{-1} g'(M) - \lambda\varepsilon^{-1} \int_{\Omega_\varepsilon^+} K'(M, N)\, v_\varepsilon(N)\, dS_N \tag{1.9}$$

where the prime denotes a derivative with respect to x or y, and its meaning is determined below.

From the theorem on differentiation of integrals with a weak singularity /18/ it follows that the second term on the right-hand side of (1.9) is a function belonging to $L_2(\Omega)$. Therefore, integrating this term by parts and taking into account the finiteness of v_ε^+, and the equations $K_x'(M,N) = -K_\xi'(M,N)$, $K_y'(M,N) = -K_\eta'(M,N)$, we obtain

$$\varepsilon v_\varepsilon' + \lambda K v_\varepsilon' = g', \quad M, N \in \Omega_\varepsilon^+ \tag{1.10}$$

From this it follows that the function v_ε^+, $M \in \Omega_\varepsilon^+$ belongs to the same class of functions as $g'(M)$, $M \in \Omega_\varepsilon^+$.

In the same way as for (1.8), (1.10), we have

$$\mu v_\varepsilon + \lambda K v_\varepsilon^+ = g, \quad \mu v_\varepsilon' + \lambda K v_\varepsilon^{+\prime} = g' \qquad M \in \Omega_\varepsilon^- \tag{1.11}$$

The first part of the lemma follows directly from (1.10), (1.11). The same equations imply that $v_\varepsilon^{+\prime} \in C(\Omega_\varepsilon^+)$ and $v_\varepsilon^{-\prime} \in C(\Omega_\varepsilon^-)$ when $g' \in C(\Omega)$ (in general case $v_\varepsilon' \notin C(\Omega)$).

The theorems which follow establish the conditions of existence (Theorem 2) and uniqueness (Theorem 3) of the solution v^* of (1.4), the continuous dependence of v^* on the vector parameter $h = (h_1, h_2, h_3)$ (Theorem 4) and some properties of v^* (Theorem 5 and its corollary)

Theorem 2. The necessary and sufficient condition for the solution $v^* \in L_2(\Omega)$ of (1.4) to exist is, that

$$\| v_\varepsilon \|_{L_2} \leqslant c, \quad \varepsilon \in (0, \varepsilon_0] \tag{1.12}$$

where the constant c is independent of ε.

Proof. Necessity. Let $v^* \in L_2(\Omega)$ be a solution of (1.4) and $S = \{M : v^* \geqslant 0\}$. Let us denote by $L_2^0(\Omega)$ the set of functions finite in Ω and belonging to $L_2(\Omega)$, and consider the following functional on the convex set $\omega = \{v \in L_2^0(\Omega), v \geqslant 0\}$ closed in L_2:

$$\varphi(v) = {}^1/_2\lambda(Kv, v) - (g, v)$$

The functional is trictly convex and $\varphi'(v) = \operatorname{grad}\varphi(v) = \lambda K v - g$. We will show that

$$\inf_{v \in \omega} \varphi(v) = \varphi(v^{*+}) \tag{1.13}$$

To do this, it is sufficient to confirm the inequality /19/

$$(\varphi'(v^{*+}), v - v^{*+}) \geqslant 0, \ \forall v \in \omega$$

or

$$(\lambda K v^{*+} - g, v) \geqslant (\lambda K v^{*+} - g, v^{*+}), \ \forall v \in \omega \tag{1.14}$$

Since $\lambda K v^{*+} - g > 0$, $v^{*+} = 0$ when $M \notin S$ and $\lambda K v^{*+} - g = 0$, $v^{*+} \geqslant 0$ when $M \in S$, it follows that the right-hand side of (1.14) is equal to zero and the left-hand side is positive, and this proves (1.13).

We can show, in the same manner, that the unique function $v_\varepsilon^+ = Qv_\varepsilon$ furnishes a minimum to the functional $\varphi_\varepsilon(v) = {}^1/_2\varepsilon(v, v) + \varphi(v)$, $v \in \omega$. Since for any $\varepsilon \in (0, \varepsilon_0]$ we have

$$\varphi_\varepsilon(v_\varepsilon^+) < {}^1/_2\varepsilon(v^{*+}, v^{*+}) + {}^1/_2\lambda(Kv^{*+}, v^{*+}) - (g, v^{*+})$$

or

$${}^1/_2\varepsilon[(v^{*+}, v^{*+}) - (v_\varepsilon^+, v_\varepsilon^+)] > \varphi(v_\varepsilon^+) - \varphi(v^{*+}) > 0$$

it follows that

$$\| v_\varepsilon^+ \|_{L_2} < \| v^{*+} \|_{L_2}, \quad \forall \varepsilon \in (0, \varepsilon_0] \tag{1.15}$$

The existence of the constant c in inequality (1.12) obviously follows from inequality (1.15).

Sufficiency. Let the inequality (1.12) hold. We will write $Av = \mu Q^- v + \lambda K Q v - g$ and form a sequence v_{ε_n} $(n = 0, 1, 2, \ldots)$ where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. We separate out of the sequences $\{v_{\varepsilon_n}\}$, $\{Qv_{\varepsilon_n}\}$, $\{Q^- v_{\varepsilon_n}\}$ weakly convergent in L_2, the subsequences (we can always do it by virtue of (1.12) and the reflexivity of the space L_2 (see /20/, p.60)), and assign to them the same notation as that of the original sequences, i.e. $v_{\varepsilon_n} \rightharpoonup v^*$, $Qv_{\varepsilon_n} \rightharpoonup w$, $Q^- v_{\varepsilon_n} \rightharpoonup u$. We shall show that v^* is a solution of (1.4).

Since $Q^- v_{\varepsilon_n} = \mu^{-1} g - \lambda\mu^{-1} K Q v_{\varepsilon_n} - \varepsilon_n \mu^{-1} Q v_{\varepsilon_n}$, inequality (1.12) holds and K is a completely continuous operator, it follows that $Q^- v_{\varepsilon_n} \to u$ /20/. Therefore

$$(Q^- v_{\varepsilon_n}, v_{\varepsilon_n}) \to (u, v^*) \tag{1.16}$$

Using the monotonicity of Q^-, we can write $(Q^- t - Q^- v_{\varepsilon_n}, t - v_{\varepsilon_n}) \geqslant 0$ for any $t \in L_2$. Passing to the limit (taking (1.16) into account), we obtain

$$(Q^- t - u, t - v^*) \geqslant 0, \ \forall t \in L_2$$

i.e. $Q^- v^* = u$ (by virtue of the continuity of Q^- and the Minti lemma /14/). From this it follows (since $Qv = v - Q^- v$) that $Qv^* = w$.

Passing to the limit in the relation

$$Av_{\varepsilon_n} \equiv \mu Q^- v_{\varepsilon_n} + \varepsilon_n Q v_{\varepsilon_n} + \lambda K Q v_{\varepsilon_n} - g - \varepsilon_n Q v_{\varepsilon_n} = -\varepsilon_n Q v_{\varepsilon_n}$$

as $v_{\varepsilon_n} \rightharpoonup v^*$, we obtain $Av^* = 0$, i.e. $v^* \in L_2$ is a solution of (1.4). This proves the theorem.

Note 1. If $v_{\varepsilon_n} \in W_p^{(1)}(\Omega)$, $p \geqslant 1$, and $\| v_{\varepsilon_n} \|_{W_p^{(1)}} \leqslant c$ (the constant c is independnet of ε_n), we can separate from the sequence v_{ε_n} (on the strength of the Sobolev inclusion theorems /17, 18/) the subsequences converging strongly to v^* in the spaces into which $W_p^{(1)}$ can be compactly imbedded. For example, when $p > 2$, the sequence will converge uniformly to v^*. The conditions of inclusion $v_\varepsilon \in W_p^{(1)}(\Omega)$ are given in the previous lemma.

Note 2. The following estimate follows direclty from (1.6):

$$\| v_\varepsilon^+ \|_{L_1} \leqslant m^{-1} \| g \|_C, \quad m = \inf_{\Omega \times \Omega} K(M, N) > 0$$

therefore

$$\lim_{\varepsilon \to \infty} \| Av_\varepsilon \|_{L_1} = \lim_{\varepsilon \to 0} \varepsilon \| v_\varepsilon^+ \|_{L_1} = 0$$

Theorem 3. If Eq.(1.4) has a solution $v^* \in L_2(\Omega)$, it is unique.

Proof. Let

$$\mu Q^- v_1 + \lambda K Q v_1 = g, \; \mu Q^- v_2 + \lambda K Q v_2 = g; \; v_1 \neq v_2 \tag{1.17}$$

We write

$$d = Qv_2 - Qv_1, \; d^- = Q^- v_2 - Q^- v_1, \; \delta = v_2 - v_1 \tag{1.18}$$

Then from (1.17) we have

$$\begin{gathered} \mu d^- + \lambda K d = 0 \\ \mu (d^-, d) + \lambda (Kd, d) = 0; \; \mu\delta - \mu d + \lambda K d = 0 \\ (d^-, d) = -(Q^- v_1, Qv_2) - (Q^- v_2, Qv_1) \geqslant 0 \end{gathered} \tag{1.19}$$

Therefore (1.19) and the strict positiveness of K together imply $d = 0, \delta = 0$. The theorem is proved.

Theorem 4. Let h be a real parameter, and $g(M) = h - f(M)$, v_h^*, $h \in [0, h_0]$ a family of solutions of (1.4) depending on h. Then v_h^* depends continuously on h and the continuous function

$$P(h) = \int_\Omega Q(v_h^*(M))\, dS_M$$

is strictly increasing ($P(h)$ is the force impressing the stamp to the depth h).

Proof. Let v_1 and v_2 be solutions of (1.4) corresponding to the values $h = h_1$ and $h = h_2$ from the interval $[0, h_0]$. Then, taking into account the notation (1.18) we have

$$\begin{gathered} \mu d^- + \lambda K d = h_2 - h_1, \; \mu\delta - \mu d + \lambda K d = h_2 - h_1 \\ \mu (d^-, d) + \lambda (Kd, d) = (h_2 - h_1, d); \; (d^-, d) \geqslant 0 \end{gathered} \tag{1.20}$$

Since $(h_2 - h_1, d) = (h_2 - h_1)(P(h_2) - P(h_1))$ and the left-hand side of the last equation of (1.20) is positive, it follows that $P(h)$ is a strictly increasing function. Further, from the last equation of (1.20) and the strict positiveness of K it follows that $d \to 0$ as $h_1 \to h_2$. Therefore the penultimate equation of (1.20) shows that $\delta \to 0$ as $h_1 \to h_2$, which implies the continuous dependence of v_h^* on h. The theorem is proved.

Note 3. We can show in the same manner the ocntinuous dependence of v_h^* on the vector parameter $h = (h_1, h_2, h_3)$ $(g(M) = h_1 + h_2 x + h_3 y - f(x, y))$. Such a continuous dependence is of essential importance when the value $h = (h_1, h_2, h_3)$ in the contact problem is determined from the condition

$$\int_\Omega Q(v_h^*(M))\, dS_M = P, \; \int_\Omega yQ(v_h^*(M))\, dS_M = M_x, \; \int_\Omega xQ(v_h^*(M))\, dS_M = M_y$$

where $\{P, M_x, M_y\}$ are given values (P is the force pressing down the stamp, and M_x, M_y are the moments acting in the stamp).

Theorem 5. Let v_1 and v_2 be solutions of (1.4) corresponding to the values $\mu = \mu_1$ and $\mu = \mu_2$, $\mu_1 \neq \mu_2$. Then $Qv_1 = Qv_2$.

Proof. We shall assume, to be specific, that $\Delta\mu = \mu_2 - \mu_1 > 0$. We have

$$\mu_1 Q^- v_1 + \lambda K Q v_1 = g, \; \mu_2 Q^- v_2 + \lambda K Q v_2 = g \tag{1.21}$$

Let us subtract the first equation of (1.21) from the second, and scalarly multiply the result by d. We obtain

$$\Delta\mu\,(Q^- v_2,\ d) + \mu_1\,(d^-,\ d) + \lambda\,(Kd,\ d) = 0 \tag{1.22}$$

We can confirm direclty that the terms on the left-hand side of (1.22) are non-negative numbers. Therefore $d = 0$, from which the assertion of the theorem follows.

Corollary. The existence of a solution of (1.4) for some value of $\mu = \mu^*$ implies the existence of solutions of (1.4) for all $\mu > 0$.

Note 4. We can deal in the same manner with equations of the type (1.4) in the case of non-symmetric kernels $K(M, N)$ (e.g. for the problems discussed in /21/ and in contact problems with frictional forces).

Thus the contact problem formulated can be recuced to solving Eq.(1.4), and various approximate methods /14—17/ can be used to achieve it. An appropriate method based on the application of a regularized boundary equation was discussed in /22/.

2. A contact problem for bodies with linear and non-linear, Winkler-type covering /9—11/. In these problems the first condition of (1.1) has the form

$$\Phi\,(u_z'\,(M)) + 2\pi\lambda u\,(M) = g\,(M)$$

Here $\Phi(t)$ $(-\infty < t < \infty)$ is a strictly increasing, continuous function of its arguement t, $\Phi(0) = 0$.

A system analogous to system (1.2) is written for the unknown pair $(w(M), S)$ thus

$$\begin{aligned} &w(M) + \lambda\int_S K(M,N)\,H(w(N))\,dS_N = g(M), \quad w(M) \geqslant 0, \quad M \in S \\ &\lambda\int_S K(M,N)\,H(w(N))\,dS_N > g(M), \quad w(M) = 0, \quad M \in (\Omega\setminus S) \end{aligned} \tag{2.1}$$

where H is a function inverse to Φ, $w(M) = \Phi(p(M))$. As before, $p(M)$ is the contact pressure and S is the area of contact. In what follows, we shall assume that the function H satisfies the condition

$$|H(w)| \leqslant c_*\,|w|^{1/\alpha};\ c_* = \text{const},\ 0 < \alpha \leqslant 1$$

Let us consider the Hammerstein-type equation for the unknown function $v(M)$

$$v(M) + \lambda\int_\Omega K(M,N)\,Q(H(v(N)))\,dS_N = g(M); \qquad M, N \in \Omega$$

which has the following operator form:

$$v + \lambda KQHv = g \tag{2.2}$$

If v^* is a solution of (2.2), then the function $w = Qv^*$ and the set $S = \{M\colon v^* \geqslant 0\}$ is a solution of system (2.1) and $S \neq \varnothing$ when $\Omega_0 \neq \varnothing$. The converse it also true. If (w, S) is a solution of (2.1), then the function $v^* = g - \lambda KHw$ $(M \in \Omega)$ is a solution of (2.2). The proof is analogous to that of Theorem 1.

Thus problem (2.1) is reduced to that of solving the Hammerstein Eq.(2.2). The uniqueness of the solution v^* of (2.2) and its continuous dependence on the parameter $h = (h_1, h_2, h_3)$ is established in the same manner as in Theorems 3 and 4. The sufficient conditions for a solution of (2.2) to exist are given in Theorem 6, where the constraints imposed on the function g are somewhat weakened (the assumption that the bounded region $\Omega_0 = \{M\colon g \geqslant 0\}$ exists is retained).

Theorem 6. Let the following conditions hold:

$$g \in L_p(\Omega),\ p = 1 + 1/\alpha,\ {}^1/_2 < \alpha \leqslant 1$$

Then Eq.(2.2) has a solution $v^* \in L_p(\Omega)$. Moreover, if $g \in C(\Omega)$, then $v^* \in C(\Omega)$.

Proof. The operator K is a completely continuous operator from L_q, $q = 1 + \alpha$, into $L_q^* = L_p$, $p = 1 + 1/\alpha$ /18/. The contraction of K on L_2 is a selfconjugate, strictly positive operator. Therefore a square root $D = K^{1/2}$ exists, which is a completely continuous operator from L_2 into L_q^* /14/. The conjugate operator D^* acts from L_q^* into L_2.

If we make the change of variable /14/ $v = Dt + g$ in (2.2), we obtain the equivalent equation

$$Ft \equiv t + \lambda D^*QH\,(Dt + g) = 0;\ t \in L_2 \tag{2.3}$$

with a continuous, monotonic and potential operator F (the monotonicity of F follows from the monotonicity of the function $QH(v)$).

Let us find a lower estimate for the scalar product (Ft, t)

$$\begin{aligned} (Ft,\ t) &= (t,\ t) + \lambda\,(QH\,(Dt + g),\ Dt + g) - \\ &\lambda\,(QH\,(Dt + g),\ g) \geqslant (t,\ t) - \lambda\,(QH\,(Dt + g),\ g\,) \geqslant \\ &(t,\ t) - \lambda\,\|g\|_{L_p}\,\|QH\,(Dt + g)\|_{L_q} \end{aligned}$$

Using the properties of the operators Q and H and the Minkowski inequality, we obtain

$$\| QH\,(Dt + g)\|_{L_q} \leqslant c_*\,(\| D \|\,\| t \|_{L_2} + \| g \|_{L_p})^{1/\alpha}$$

Therefore, we have the following estimate:

$$(Ft, t) \geqslant \| t \|_{L_2}{}^2 - c_*\lambda\,\| g \|_{L_p}\,\| t \|_{L_2}^{1/\alpha} \left(\| D \| + \frac{\| g \|_{L_p}}{\| t \|_{L_2}}\right)^{1/\alpha}$$

and a number $\rho > 0$ exists when $\alpha > 1/2$ such, that when $\| t \|_{L_2} \geqslant \rho$, the inequality $(Ft, t) > 0$ holds, i.e. according to the Brauder-Minti theorem /14/ Eq.(2.3) has a solution $t^* \in L_2$ and (2.2) has the corresponding solution $v^* = Dt^* + g$.

We shall show now that $v^* \in C\,(\Omega)$ when $g \in C\,(\Omega)$. Let $M \in S = \{M\colon v^* \geqslant 0\}$ (the existence of $S \neq \varnothing$ is shown in exactly the same manner as in Theorem 1). Then we have, for $M \in S$,

$$v^*(M) + \lambda \int_S K\,(M, N)\,H\,(v^*(N))\,dS_N = g\,(M) \tag{2.4}$$

and the following alternative is possible: 1) $v^*(M)$ $(M \in S)$ is a discontinuous bounded function, and 2) $v^*(M)$ $(M \in S)$ is an unbounded function.

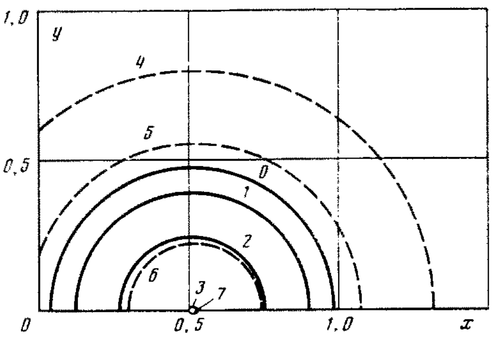

Fig.1

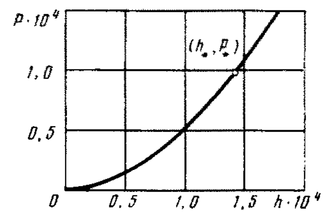

Fig.2

In the first case the left-hand side of (2.4) is a discontinuous bounded function (since λKHv^* is a continuous function), and this contradicts the continuity of g. In the second case the left-hand side of (2.4) is an unbounded function (since it is a sum of non-negative functions of which at least one is unbounded). This again contradicts the continuity of g. Therefore $v^*(M)$ $(M \in S)$ is a continuous function. This, and the properties of the potential of a simple layer imply that the function $v^* = g - \lambda KQHv^*$ $(M \in \Omega)$ is continuous, and this proves Theorem 6.

Note 5. If $g \in C\,(\Omega)$, then the operator U, completely continuous in $C\,(\Omega)$ defined by the relation $Uv = g - \lambda KQHv$, maps the segment $[Ug, g] \subset C\,(\Omega)$ onto itself (the constraint $|H\,(w)| \leqslant c_*\,|w|^{1/\alpha}$ can be omitted here). Therefore the Schauder principle /17/ implies at once that Eq. (2.2) has a solution $v^* \in [Ug, g]$ for all $\lambda > 0$.

Note 6. When $\mu = \varepsilon$, Eq.(1.6) is of the type (2.2).

The method discussed here of studying contact problems using boundary, Hammerstein-type equations, is fairly general. In many contact problems (e.g. in problems dealing with contact between plates and beams with an elastic foundation, and in problems of contact between rough bodies), the conditions of contact between the bodies for which Green's function is known i.e. the response of each body to a unit excitation), can be reduced with help of the operator Q to a Hammerstein-type equation.

Numerical example. Eq.(1.2) was solved with $H = E$ using the method of successive approximations

$$v_{n+1} = g - \lambda KQv_n;\quad v_0 = g;\quad n = 0, 1, 2, \ldots \tag{2.5}$$

for the following data:

$$g\,(M) = h - f\,(x, y)$$
$$f\,(x, y) = \begin{cases} (2R)^{-1}\,((x - a)^2 + y^2), & x \geqslant 0 \\ (2R)^{-1}\,((x + a)^2 + y^2), & x < 0 \end{cases}$$
$$\Omega = \{-1{,}5 \leqslant x \leqslant 1{,}5;\ -1{,}0 \leqslant y \leqslant 1{,}0\},\quad 0 \leqslant h \leqslant 0{,}5\cdot 10^{-3}$$

which correspond to the problem of imbedding a stamp with Winkler-type covering, to a depth h into an elastic half-space. The stamp consists of "paired" paraboloids of rotation whose apices are separated by a distance $2a\ (a > 0)$.

Eq.(2.5) was discretized, remembering that the solution v^* is symmetric about the x and y axes. The nodes of the mesh approximating the rectangle Ω, have the following coordinates:

$$x = ih_x, \quad y = jh_y; \quad i, \ j = 0, \ \pm 1, \pm 2, \ldots, \pm m$$

Here $h_x = 1{,}5/m$, $h_y = 1.0/m$ are the mesh steps in the x and y direction, respectively. The operator K was approximated as in /22/ using the rectangle formulas. Process (2.5) was terminated according to the criterion $\| v_{n+1} - v_n \| / \| v_n \| \leqslant \varepsilon$, $\varepsilon = 10^{-3}$.

Fig.1 shows the isobars $p(M) \cdot 10^5 = c = \text{const}$ for the following values of the parameters: $\lambda = 0{,}07$; $R = 10^3$; $a = 0{,}5$; $m = 10$. The solid lines $0, 1, 2, 3$ have the corresponding values $c = 0$; $3{,}46$; $7{,}96$; 10.64 and $h = 1{,}25 \cdot 10^{-4}$ (the region S of contact is doubly connected and the force pressing down the stamp is $P = 7{,}9 \cdot 10^{-5}$). The dashed lines $4, 5, 6, 7$ have the corresponding values $c = 0$; $16{,}5$; $26{,}7$; $28{,}9$ and $h = 3{,}75 \cdot 10^{-4}$ (the area S of contact is singly connected, $P = 5{,}7 \cdot 10^{-4}$).

Fig.2 shows the function $P = P(h)$. The point (h_*, P_*) corresponds to the passage from the doubly connected region of contact to the singly connected one.

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